

STOCHASTIC LAGRANGIAN PARTICLE APPROACH TO FRACTAL NAVIER-STOKES EQUATIONS

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ABSTRACT. In this article we study the fractal Navier-Stokes equations by using stochastic Lagrangian particle path approach in Constantin and Iyer [6]. More precisely, a stochastic representation for the fractal Navier-Stokes equations is given in terms of stochastic differential equations driven by Lévy processes. Basing on this representation, a self-contained proof for the existence of local unique solution for the fractal Navier-Stokes equation with initial data in $\mathbb{W}^{1,p}$ is provided, and in the case of two dimensions or large viscosity, the existence of global solution is also obtained. In order to obtain the global existence in any dimensions for large viscosity, the gradient estimates for Lévy processes with time dependent and discontinuous drifts is proved.

1. INTRODUCTION

Consider the following incompressible fractal or generalized Navier-Stokes equation in \mathbb{R}^d (abbreviated as FNSE):

$$\begin{cases} \partial_t u = \mathcal{L}u - (u \cdot \nabla)u + \nabla p, & t \geq 0, \\ \nabla \cdot u = 0, & u(0) = u_0, \end{cases} \quad (1.1)$$

where $u = (u^1, \dots, u^d)^t$ denotes the column vector of velocity field, p is the pressure, \mathcal{L} is the generator of a Lévy process given by

$$\mathcal{L}u(x) = \int_{\mathbb{R}^d} (u(x+y) - u(x) - 1_{|y| \leq 1} (y \cdot \nabla)u(x)) \nu(dy), \quad (1.2)$$

where ν is a Lévy measure on \mathbb{R}^d , i.e., it satisfies that $\nu\{0\} = 0$ and

$$\int_{\mathbb{R}^d} 1 \wedge |y|^2 \nu(dy) < +\infty.$$

When $\nu(dy) = dy/|y|^{d+\alpha}$ with $\alpha \in (0, 2)$, $\mathcal{L} = -c_\alpha(-\Delta)^{\alpha/2}$ is the usual fractional Laplacian operator by multiplying a constant.

As a simplified model of equation (1.1), the following fractal Burgers equation has been studied by Biler, Funaki and Woyczynski [3] and Kiselev, Nazarov and Schterenberg [15],

$$\partial_t u = -(-\Delta)^{\alpha/2} u - (u \cdot \nabla)u, \quad t \geq 0, \quad u(0) = u_0.$$

As for generalized Navier-Stokes equation (1.1), when $\mathcal{L} = -(-\Delta)^{\alpha/2}$, it has been studied by Wu [25] in Besov spaces by using purely analytic argument. The main feature of such fractal equations is that operator \mathcal{L} given by (1.2) is non-local. Recently, there are increasing interests for studying such fractal equations or fractional dissipative equations since they naturally appear in hydrodynamics, statistical mechanics, physiology, certain combustion models, and so on (cf. [21, 19, 24], etc.).

The aim of this paper is to study equation (1.1) by using a stochastic Lagrangian particle trajectories approach following [6, 27]. More precisely, Constantin and Iyer [6] gave the following elegant stochastic representation for the regularity solution u of Navier-Stokes equation (corresponding to $\mathcal{L} = \nu\Delta$ in (1.1)):

$$\begin{cases} X_t(x) = x + \int_0^t u_s(X_s(x))ds + \sqrt{2\nu}B_t, & t \geq 0, \\ u_t = \mathbf{P}\mathbb{E}[(\nabla^t X_t^{-1})(u_0 \circ X_t^{-1})], \end{cases} \quad (1.3)$$

where \mathbf{P} denotes the Leray-Hodge projection onto divergence free vector fields, B_t is a Brownian motion, and $X_t^{-1}(x)$ denotes the inverse of $x \mapsto X_t(x)$. Basing on this representation, a self-contained proof of the existence of local smooth solutions in Hölder space was given by Iyer [13]. Later on, by reversing the time variable, in a previous work [27], we considered the following stochastic representation:

$$\begin{cases} X_{t,s}(x) = x + \int_t^s u_r(X_{t,r}(x))dr + \sqrt{2\nu}(B_s - B_t), & t \leq s \leq 0, \\ u_t = \mathbf{P}\mathbb{E}[(\nabla^t X_{t,0})(u_0 \circ X_{t,0})], \end{cases} \quad (1.4)$$

and a self-contained proof of the existence of local smooth solutions in Sobolev space is also obtained. Moreover, the global solution for large viscosity is proven by using Bismut formula.

Naturally, if one replaces the Brownian motion in (1.4) by a general Lévy process L_t , then it is expected that the corresponding solution u will solve the following backward fractal Navier-Stokes equation:

$$\begin{cases} \partial_t u + \mathcal{L}u + (u \cdot \nabla)u + \nabla p = 0, & t \leq 0, \\ \nabla \cdot u = 0, & u(0) = u_0, \end{cases} \quad (1.5)$$

where \mathcal{L} is the Lévy generator of the Lévy process L_t . In general, it seems hard to solve the above fractal Navier-Stokes equation by using purely analytic tools. However, stochastic system (1.4) is easier to be dealt with if one replaces B_t by a general process and only considers the local smooth solutions. In fact, it is easy to obtain the existence of local smooth solutions for stochastic system (1.4), and a global smooth solution in two dimensional case by the same arguments as in [13, 28]. This will be given in Section 2.

On the other hand, if we only assume that the initial data belongs to the first order Sobolev space $\mathbb{W}^{1,p}$, it seems not so easy to construct a local solution in $\mathbb{W}^{1,p}$ for stochastic system (1.4). A clear difficulty is to obtain the differentiability of the solution flow $x \mapsto X_{t,0}(x)$. Although one can solve the following equation with divergence free vector field $u \in L_{loc}^1((-\infty, 0]; \mathbb{W}^{1,p})$ by using DiPerna-Lions' theory (cf. [10, 7]):

$$X_{t,s}(x) = x + \int_t^s u_r(X_{t,r}(x))dr + (L_s - L_t),$$

it is only known that $x \mapsto X_{t,0}(x)$ is approximately differentiable (cf. [1, 7]). This difficulty will be overcome by using Krylov's estimate for jump-diffusion processes and the regularizing effect of Lévy process if L_t is non-degenerated in some sense (see Condition $(\mathbf{H})_\alpha$ below). Thus, following Section 2, Section 3 will be devoted to the proof of the existence of local $\mathbb{W}^{1,p}$ -solutions.

For proving the existence of global solutions in $\mathbb{W}^{1,p}$ for large viscosity, we need some gradient estimates for the SDE with Sobolev coefficients and driven by a Lévy process. For this aim, we shall use some asymptotic estimates for the heat kernels of Lévy processes due to Schilling, Sztonyk and Wang [20]. Our approach for the gradient estimates of SDEs is based on the a priori estimate for an integro-differential equation and the uniqueness of weak solutions. This

is the content of Section 4, and can be read independently. In Section 5, we prove the global well posedness for large viscosity.

Lastly, we mention that other stochastic approaches for incompressible Navier-Stokes equations can be found in the references [16, 4, 5, 8, 9], etc.; and compared with the analytic arguments, one of the main advantages of representation (1.4) is that it is convenient for numerical simulations (cf. [17, 14]). This is in fact our main motivation for studying the stochastic representation of fractal Navier-Stokes equation (1.5).

2. STOCHASTIC REPRESENTATION FOR FRACTAL NAVIER-STOKES EQUATIONS

We first fix some notations. Set $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ and $\mathbb{R}_- := (-\infty, 0]$. For $k \in \mathbb{N}_0$, let $C_b^k = C_b^k(\mathbb{R}^d; \mathbb{R}^d)$ be the space of all k -order continuously differentiable vector fields on \mathbb{R}^d with the norm

$$\|u\|_{C_b^k} := \sum_{j=0}^k \sup_{x \in \mathbb{R}^d} |\nabla^j u(x)| < +\infty,$$

where ∇^j denotes the j -order gradient, and $|\cdot|$ denotes the Euclidean norm. For $k \in \mathbb{N}_0$ and $p \geq 1$, let $\mathbb{W}^{k,p} = \mathbb{W}^{k,p}(\mathbb{R}^d; \mathbb{R}^d)$ be the usual vector-valued Sobolev space on \mathbb{R}^d with the norm

$$\|u\|_{k,p} := \sum_{j=0}^k \|\nabla^j u\|_p < +\infty,$$

where $\|\cdot\|_p$ is the usual L^p -norm in \mathbb{R}^d .

Let us now recall some basic notions and facts about Lévy processes on negative time axis. Let $(L_t)_{t \leq 0}$ be an \mathbb{R}^d -valued Lévy process on some probability space (Ω, \mathcal{F}, P) , i.e., an \mathbb{R}^d -valued stochastically continuous process with stationary independent increments and $L_0 = 0$. By Lévy-Khintchine's formula (cf. [2, p.109, Corollary 2.4.20]), the characteristic function of L_t is given by

$$\mathbb{E}(e^{i\xi \cdot L_t}) = \exp \left\{ t \left[ib \cdot \xi + \xi^t A \xi + \int_{\mathbb{R}^d} [1 - e^{i\xi \cdot x} + i\xi \cdot x 1_{|x| \leq 1}] \nu(dx) \right] \right\} =: e^{t\psi(\xi)}, \quad (2.1)$$

where $\psi(\xi)$ is a complex-valued function called the symbol of $(L_t)_{t \leq 0}$, and $b \in \mathbb{R}^d$, $A \in \mathbb{R}^d \times \mathbb{R}^d$ is a positive definite and symmetric matrix, ν is a Lévy measure on \mathbb{R}^d . Throughout this paper, we only consider the pure jump Lévy process, and assume below that

$$b = 0, \quad A = 0.$$

We remark that $t \mapsto L_t$ admits a version still denoted by L_t such that for almost all ω , $t \mapsto L_t(\omega)$ is right continuous and has left limit, but, for fixed t ,

$$P\{\omega : L_t(\omega) \neq L_{t-}(\omega)\} = 0.$$

Below, for $t \leq s \leq 0$, define

$$\mathcal{F}_{t,s} := \sigma\{L_r - L_t : t \leq r \leq s\}. \quad (2.2)$$

Given $u \in C(\mathbb{R}_-; C_b^3(\mathbb{R}^d; \mathbb{R}^d))$, for $x \in \mathbb{R}^d$, let $X_{t,s}(x)$ be the unique solution of the following SDE:

$$X_{t,s}(x) = x + \int_t^s u_r(X_{t,r}(x)) dr + (L_s - L_t), \quad t \leq s \leq 0. \quad (2.3)$$

It is easy to see that $\{X_{t,s}(x), x \in \mathbb{R}^d, t \leq s \leq 0\}$ forms a stochastic C^3 -diffeomorphism flow, and

$$\nabla X_{t,s}(x) = \mathbb{I} + \int_t^s \nabla u_r(X_{t,r}(x)) \cdot \nabla X_{t,r}(x) dr, \quad (2.4)$$

where $\nabla X_{t,s}(x) = (\partial_j X_{t,s}^i(x))_{i,j=1,\dots,d}$, and ∂_j denotes the partial derivative with respect to the j -th component of x .

Let $N(t, \Gamma) := \sum_{t \leq s < 0} 1_\Gamma(L_s - L_{s-})$, $\Gamma \in \mathcal{B}(\mathbb{R}^d)$ be the Poisson random point measure associated with $(L_t)_{t \leq 0}$. By Lévy-Itô's decomposition (cf. [2, p.108, Theorem 2.4.16]), one has

$$L_t = \int_{|x| \leq 1} x \tilde{N}(t, dx) + \int_{|x| > 1} x N(t, dx),$$

where $\tilde{N}(t, dx) := N(t, dx) - t\nu(dx)$ is the compensated random martingale measure. For $g \in C_b^2(\mathbb{R}^d; \mathbb{R})$, by Itô's formula (cf. [2, p.226, Theorem 4.4.7]), we have

$$g(X_{t,s}) = g(x) + \int_t^s [\mathcal{L}g(X_{t,r}) + (u_r \cdot \nabla)g(X_{t,r})]dr + M_{t,s}^g, \quad (2.5)$$

where \mathcal{L} is the generator of $(L_t)_{t \leq 0}$ given by (1.2), and $(M_{t,s}^g)_{s \in [t,0]}$ is a square integrable martingale given by

$$M_{t,s}^g := \int_t^s \int_{\mathbb{R}^d} [g(X_{t,r-} + y) - g(X_{t,r-})] \tilde{N}(dr, dy).$$

We have

Theorem 2.1. *Let $u \in C(\mathbb{R}_-; C_b^3(\mathbb{R}^d; \mathbb{R}^d))$ and $X_{t,s}(x)$ be the solution of SDE (2.3). For $\varphi \in C_b^2(\mathbb{R}^d; \mathbb{R}^d)$ and $c \in C(\mathbb{R}_-; C_b^2(\mathbb{R}^d; \mathbb{R}))$, define*

$$h_t(x) := \mathbb{E} \left[\exp \left\{ \int_t^0 c_r(X_{t,r}(x)) dr \right\} \varphi(X_{t,0}(x)) \right],$$

and

$$w_t(x) := \mathbb{E}[\nabla^t X_{t,0}(x) \cdot \varphi(X_{t,0}(x))].$$

Then $h, w \in C^1(\mathbb{R}_-; C_b^2(\mathbb{R}^d; \mathbb{R}^d))$ respectively and uniquely solve the following partial integro-differential equations (PIDE):

$$\partial_t h_t + \mathcal{L}h_t + (u_t \cdot \nabla)h_t + c_t h_t = 0, \quad h_0(x) = \varphi(x), \quad (2.6)$$

and

$$\partial_t w_t + \mathcal{L}w_t + (u_t \cdot \nabla)w_t + (\nabla^t u_t)w_t = 0, \quad w_0(x) = \varphi(x). \quad (2.7)$$

Proof. Fix $t < 0$. For $g \in C_b^2(\mathbb{R}^d)$ and $\delta > 0$, by Itô's formula (see (2.5)), we have

$$\begin{aligned} & \mathbb{E} \left[\exp \left\{ \int_{t-\delta}^t c_r(X_{t-\delta,r}(x)) dr \right\} g(X_{t-\delta,t}(x)) \right] - g(x) \\ &= \mathbb{E} \left[\int_{t-\delta}^t \exp \left\{ \int_{t-\delta}^s c_r(X_{t-\delta,r}(x)) dr \right\} [\mathcal{L}g(X_{t-\delta,s}(x)) + (u_s \cdot \nabla)g(X_{t-\delta,s}(x))] ds \right] \\ &+ \mathbb{E} \left[\int_{t-\delta}^t \exp \left\{ \int_{t-\delta}^s c_r(X_{t-\delta,r}(x)) dr \right\} c_s(X_{t-\delta,s}(x)) g(X_{t-\delta,s}(x)) ds \right] \end{aligned}$$

By the stochastic continuity of $t \mapsto L_t$, from equation (2.3), it is easy to prove that $(t, s) \rightarrow X_{t,s}(x)$ is also stochastically continuous. Thus, since $(s, x) \mapsto \mathcal{L}g(x) + (u_s \cdot \nabla)g(x) + c_s(x)g(x)$ is bounded and continuous, by the dominated convergence theorem, we have

$$\begin{aligned} & \frac{1}{\delta} \left[\mathbb{E} \left[\exp \left\{ \int_{t-\delta}^t c_r(X_{t-\delta,r}(x)) dr \right\} g(X_{t-\delta,t}(x)) \right] - g(x) \right] \\ & \xrightarrow{\delta \rightarrow 0} \mathcal{L}g(x) + (u_t \cdot \nabla)g(x) + c_t(x)g(x). \end{aligned} \quad (2.8)$$

Noticing that

$$X_{t-\delta,0}(x) = X_{t,0} \circ X_{t-\delta,t}(x),$$

by Markov property, we have

$$\begin{aligned}
h_{t-\delta} &= \mathbb{E} \left[\exp \left\{ \int_{t-\delta}^t c_r(X_{t-\delta,r}) dr \right\} \exp \left\{ \int_t^0 c_r(X_{t,r} \circ X_{t-\delta,t}) dr \right\} \varphi(X_{t,0} \circ X_{t-\delta,t}) \right] \\
&= \mathbb{E} \left[\exp \left\{ \int_{t-\delta}^t c_r(X_{t-\delta,r}) dr \right\} \mathbb{E} \left[\exp \left\{ \int_t^0 c_r(X_{t,r} \circ X_{t-\delta,t}) dr \right\} \varphi(X_{t,0} \circ X_{t-\delta,t}) \middle| \mathcal{F}_{t-\delta,t} \right] \right] \\
&= \mathbb{E} \left[\exp \left\{ \int_{t-\delta}^t c_r(X_{t-\delta,r}) dr \right\} h_t \circ X_{t-\delta,t} \right].
\end{aligned}$$

Thus, by (2.8), we obtain

$$\frac{1}{\delta}(h_t(x) - h_{t-\delta}(x)) \xrightarrow{\delta \rightarrow 0} -[\mathcal{L}h_t(x) + (u_t \cdot \nabla)h_t(x) + c_t(x)h_t(x)].$$

Since the limit is a continuous function of (t, x) , it follows that for each x , $t \mapsto h_t(x)$ is differentiable and equation (2.6) is obtained.

As for (2.7), observing that

$$\nabla X_{t-\delta,0}(x) = (\nabla X_{t,0}) \circ X_{t-\delta,t}(x) \cdot \nabla X_{t-\delta,t}(x),$$

by Markov property again, we have

$$w_{t-\delta}(x) = \mathbb{E}[\nabla^t X_{t-\delta,0}(x) \cdot \varphi(X_{t-\delta,0}(x))] = \mathbb{E}[\nabla^t X_{t-\delta,t}(x) \cdot w_t(X_{t-\delta,t}(x))].$$

Hence, by (2.8) and (2.4), we have

$$\begin{aligned}
\frac{1}{\delta}(w_t(x) - w_{t-\delta}(x)) &= -\frac{1}{\delta}\mathbb{E}[w_t(X_{t-\delta,t}(x)) - w_t(x)] - \frac{1}{\delta}\mathbb{E}[(\nabla^t X_{t-\delta,t} - \mathbb{I}) \cdot w_t(X_{t-\delta,t}(x))] \\
&\xrightarrow{\delta \rightarrow 0} -[\mathcal{L}w_t + (u_t \cdot \nabla)w_t](x) - (\nabla^t u_t \cdot w_t)(x).
\end{aligned}$$

Equation (2.7) is thus obtained.

We now prove the uniqueness. Here we adopt the duality argument. Let $\hat{X}_{t,s}(x)$ solve the following SDE:

$$\hat{X}_{t,s}(x) = x - \int_t^s u_r(\hat{X}_{t,r}(x)) dr - (L_s - L_t), \quad t \leq s \leq 0.$$

Fix $\phi \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d)$ and $T < 0$. For $t \in [T, 0]$, define

$$\hat{h}_t(x) := \mathbb{E} \left[\exp \left\{ \int_{T-t}^0 (c_r - \operatorname{div} u_r)(X_{T-t,r}(x)) dr \right\} \phi(X_{T-t}(x)) \right].$$

By the above proof, it follows that $\hat{h}_t \in L^1(\mathbb{R}^d) \cap C_b^2(\mathbb{R}^d)$ solves the following PIDE:

$$\begin{aligned}
\partial_t \hat{h}_t &= \mathcal{L}^* \hat{h}_t - (u_t \cdot \nabla) \hat{h}_t + (c_t - \operatorname{div} u_t) \hat{h}_t \\
&= \mathcal{L}^* \hat{h}_t - \operatorname{div}(u_t \otimes \hat{h}_t) + c_t \hat{h}_t
\end{aligned}$$

subject to $\hat{h}_T(x) = \phi(x)$, where \mathcal{L}^* is the dual operator of \mathcal{L} and given by

$$\mathcal{L}^* g(x) = \int_{\mathbb{R}^d} [g(x-y) - g(x) + (y \cdot \nabla)g(x) 1_{|y| \leq 1}] \nu(dy).$$

Now, let $h \in C^1(\mathbb{R}_-; C_b^2(\mathbb{R}^d; \mathbb{R}^d))$ solve (2.6) with $h_0(x) \equiv 0$. Then, by the integration by parts formula, we have

$$\partial_t \langle h_t, \hat{h}_t \rangle = -\langle \mathcal{L}h_t + (u_t \cdot \nabla)h_t + c_t h_t, \hat{h}_t \rangle + \langle h_t, \mathcal{L}^* \hat{h}_t - \operatorname{div}(u_t \otimes \hat{h}_t) + c_t \hat{h}_t \rangle = 0,$$

where $\langle h_t, \hat{h}_t \rangle = \int_{\mathbb{R}^d} \langle h_t(x), \hat{h}_t(x) \rangle_{\mathbb{R}^d} dx$. Since $\langle h_0, \hat{h}_0 \rangle = 0$, it is immediate that $\langle h_T, \hat{h}_T \rangle = \langle h_T, \phi \rangle = 0$, which then gives $h_T(x) = 0$ by the arbitrariness of $\phi \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d)$. \square

Remark 2.2. If one assumes that $u \in L^1_{loc}(\mathbb{R}_-; C^3_b(\mathbb{R}^d; \mathbb{R}^d))$ and $c \in L^1_{loc}(\mathbb{R}_-; C^2_b(\mathbb{R}^d; \mathbb{R}))$, then the conclusions of Theorem 2.1 still hold if one replaces equations (2.6) and (2.7) by

$$h_t(x) = \varphi(x) + \int_t^0 [\mathcal{L}h_s(x) + (u_s \cdot \nabla)h_s(x) + c_s(x)h_s(x)]ds,$$

and

$$w_t(x) = \varphi(x) + \int_t^0 [\mathcal{L}w_s(x) + (u_s \cdot \nabla)w_s(x) + (\nabla^t u_s)w_s(x)]ds.$$

Using Theorem 2.1, we have the following representation for the solution of fractal Navier-Stokes equation as in [27, Theorem 2.3].

Theorem 2.3. Let $u \in C(\mathbb{R}_-; C^3_b(\mathbb{R}^d; \mathbb{R}^d))$ be divergence free. Then, u is a solution of fractal Navier-Stokes equation (1.5) if and only if u solves the following stochastic system:

$$\begin{cases} X_{t,s}(x) = x + \int_t^s u_r(X_{t,r}(x))dr + (L_s - L_t), & t \leq s \leq 0, \\ u_t = \mathbf{P}\mathbb{E}[\nabla^t X_{t,0} \cdot (u_0 \circ X_{t,0})], \end{cases} \quad (2.9)$$

where L is a Lévy process with generator \mathcal{L} , and \mathbf{P} is the Leray-Hodge projection onto divergence free vector fields.

Along the completely same lines as in [27, Theorems 3.8 and 4.2], one has the following result. The details are omitted.

Theorem 2.4. For any $k \in \mathbb{N}_0$ and $p > d$, there exists a constant $C_0 = C_0(k, p, d) > 0$ such that for any $u_0 \in \mathbb{W}^{k+2,p}(\mathbb{R}^d; \mathbb{R}^d)$ divergence free and $T := -(C_0 \|\nabla u_0\|_{k+1,p})^{-1}$, there is a unique pair of (u, X) with $u \in C([T, 0]; \mathbb{W}^{k+2,p})$ satisfying (2.9). Moreover, for any $t \in [T, 0]$,

$$\|\nabla u_t\|_{k+1,p} \leq C_0 \|\nabla u_0\|_{k+1,p}.$$

In two dimensional case, one has that for all $t \in \mathbb{R}_-$,

$$\|u_t\|_{k+2,p} \leq C(\|u_0\|_{k+2,p}, k, p, t),$$

where the constant C is increasing with respect to its first argument. In particular, there exists a unique global solution $u \in C(\mathbb{R}_-; \mathbb{W}^{k+2,p})$ to (2.9) in the two dimensional case.

3. EXISTENCE OF LOCAL SOLUTIONS FOR FNSE WITH $\mathbb{W}^{1,p}$ INITIAL DATA

In the remaining sections, we mainly study equation (2.9) with $u_0 \in \mathbb{W}^{1,p}(\mathbb{R}^d; \mathbb{R}^d)$. For this aim, we assume that

(H) $_{\alpha}$ Let $\psi(\xi)$ be the Lévy symbol given in (2.1) and satisfy that for some $\alpha \in (0, 2)$,

$$\operatorname{Re}(\psi(\xi)) \asymp |\xi|^{\alpha} \text{ as } |\xi| \rightarrow \infty,$$

where $a \asymp b$ means that for some $c_1, c_2 > 0$, $c_1 b \leq a \leq c_2 b$.

Consider the following SDE:

$$X_{t,s}(x) = x + \int_t^s u_r(X_{t,r}(x))dr + \nu^{1/\alpha}(L_s - L_t), \quad t \leq s \leq 0, \quad (3.1)$$

where $u : \mathbb{R}_- \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a bounded Borel measurable function, and with a little abuse of notations, $\nu \geq 0$ denotes a positive constant which plays the viscosity role.

We recall the following Krylov estimate for jump diffusion processes taken from [28, Theorem 3.7]. Although the theorem is given therein only for α -stable processes, it is clearly also valid for more general Lévy processes considered in the present paper since the proof only depends on the gradient estimate (4.5) below.

Theorem 3.1. Suppose that $(\mathbf{H})_\alpha$ holds with $\alpha \in (1, 2)$, and u is bounded by κ . Let $X_{t,s}(x)$ solve equation (3.1). Then for any $p > \frac{d}{\alpha}$ and $q > \frac{p\alpha}{p\alpha-d}$, there exists a constant $C_\kappa = C_\kappa(d, \alpha, p, q, \psi) > 0$ independent of $\nu \geq 1$ and $x \in \mathbb{R}^d$ such that for all $-1 \leq t \leq s_1 < s_2 \leq 0$ and $f \in L^q([s_1, s_2]; L^p(\mathbb{R}^d))$,

$$\mathbb{E} \left(\int_{s_1}^{s_2} f_r(X_r(x)) dr \middle| \mathcal{F}_{t,s_1} \right) \leq C_\kappa \|f\|_{L^q([s_1, s_2]; L^p(\mathbb{R}^d))}, \quad (3.2)$$

where C_κ is increasing with respect to κ , and \mathcal{F}_{t,s_1} is defined by (2.2).

The following lemma is taken from [11, p. 1, Lemma 1.1].

Lemma 3.2. Fix $t < 0$. Let $\{\beta(s)\}_{s \in [t, 0]}$ be a nonnegative measurable $(\mathcal{F}_{t,s})$ -adapted process. Assume that for all $t \leq s_1 \leq s_2 \leq 0$,

$$\mathbb{E} \left(\int_{s_1}^{s_2} \beta(r) dr \middle| \mathcal{F}_{t,s_1} \right) \leq \rho(s_1, s_2),$$

where $\rho(s_1, s_2)$ is a nonrandom interval function satisfying the following conditions:

- (i) $\rho(s_1, s_2) \leq \rho(s_3, s_4)$ if $(s_1, s_2) \subset (s_3, s_4)$;
- (ii) $\lim_{\delta \downarrow 0} \sup_{t \leq s_1 \leq s_2 \leq 0, |s_1 - s_2| \leq \delta} \rho(s_1, s_2) = 0$.

Then for any $\gamma > 0$,

$$\mathbb{E} \exp \left\{ \gamma \int_t^0 \beta(r) dr \right\} \leq 2^N,$$

where $N \in \mathbb{N}$ is chosen being such that for any $k = 0, \dots, N-1$,

$$\rho(-(k+1)|t|/N, -k|t|/N) \leq \frac{1}{2\gamma}.$$

Let f be a locally integrable function on \mathbb{R}^d . The Hardy-Littlewood maximal function is defined by

$$\mathcal{M}f(x) := \sup_{r>0} \frac{1}{|B_r|} \int_{B_r} f(x+y) dy,$$

where $B_r := \{y \in \mathbb{R}^d : |y| < r\}$ and $|B_r|$ is the volume of B_r .

We recall the following well known results (cf. [18, Appdenix] and [22, p. 5, Theorem 1]).

Lemma 3.3. (i) For any $f \in \mathbb{W}^{1,p}$, there exist $C_d > 0$ and a null set E such that for all $x, y \notin E$,

$$|f(x) - f(y)| \leq C_d (\mathcal{M}|\nabla f|(x) + \mathcal{M}|\nabla f|(y)) |x - y|. \quad (3.3)$$

(ii) For any $p > 1$, there exists a constant $C_{d,p} > 0$ such that for any $f \in L^p(\mathbb{R}^d)$,

$$\|\mathcal{M}f\|_p \leq C_{d,p} \|f\|_p. \quad (3.4)$$

Using the above three tools, we can derive the following important estimates for later use.

Lemma 3.4. Suppose that $(\mathbf{H})_\alpha$ holds with $\alpha \in (1, 2)$, and $p > \frac{2d}{\alpha}$. For any $U > 0$, there exists a time $T = T(U) \in [-1, 0)$ independent of $\nu \geq 1$ such that for any divergence free $u \in L^\infty([T, 0]; \mathbb{W}^{1,p}(\mathbb{R}^d; \mathbb{R}^d))$ with

$$\sup_{t \in [T, 0]} \|u_t\|_{1,p} \leq U, \quad (3.5)$$

the unique solution $X_{t,s}(x)$ to SDE (3.1) belongs to $\cap_{\gamma \geq 1} \mathbb{W}_{loc}^{1,\gamma}$ with respect to x , and preserves the volume, and satisfies that for any $\gamma \geq 1$ and some $C = C(T, \gamma, U) > 0$,

$$\sup_{t \in [T, 0]} \sup_{x \in \mathbb{R}^d} \mathbb{E} |\nabla X_{t,0}(x)|^\gamma \leq C, \quad (3.6)$$

and

$$\sup_{t \in [T, 0]} \sup_{x \in \mathbb{R}^d} \mathbb{E} |\nabla X_{t,0}(x)|^4 \leq 2. \quad (3.7)$$

Moreover, for any $\varphi \in \mathbb{W}^{1,p}(\mathbb{R}^d; \mathbb{R}^d)$, if we define

$$w_t := \mathbf{PE}(\nabla^t X_{t,0} \cdot (\varphi \circ X_{t,0})), \quad (3.8)$$

then $w \in C([T, 0]; \mathbb{W}^{1,p})$ and

$$\partial_t w_t = \mathbf{PE}[\nabla^t X_{t,0} \cdot (\nabla \varphi - \nabla^t \varphi) \circ X_{t,0} \cdot \partial_t X_{t,0}]. \quad (3.9)$$

Proof. Under (3.5), it has been proven in [28, Theorem 1.1] (see also [12]) that SDE (3.1) admits a unique strong solution $X_{t,s}(x)$ for each $x \in \mathbb{R}^d$. Since u is divergence free, $x \mapsto X_{t,x}(x)$ preserves the volume. Let $u_t^\varepsilon(x) := u_t * \rho_\varepsilon(x)$ be the mollifying approximation of u , where $(\rho_\varepsilon)_{\varepsilon \in (0,1)}$ is a family of mollifiers. Let $X_{t,s}^\varepsilon(x)$ solve the following SDE

$$X_{t,s}^\varepsilon(x) = x + \int_t^s u_r^\varepsilon(X_{t,r}^\varepsilon(x)) dr + \nu^{1/\alpha}(L_s - L_t), \quad t \leq s \leq 0.$$

Then

$$\nabla X_{t,s}^\varepsilon(x) = \mathbb{I} + \int_t^s \nabla u_r^\varepsilon(X_{t,r}^\varepsilon(x)) \cdot \nabla X_{t,r}^\varepsilon(x) dr,$$

and

$$|\nabla X_{t,s}^\varepsilon(x)| \leq 1 + \int_t^s |\nabla u_r^\varepsilon(X_{t,r}^\varepsilon(x))| \cdot |\nabla X_{t,r}^\varepsilon(x)| dr,$$

where $|\cdot|$ denotes the Hilbert-Schmidt norm for a matrix. By Gronwall's inequality,

$$|\nabla X_{t,s}^\varepsilon(x)| \leq \exp \left\{ \int_t^s |\nabla u_r^\varepsilon(X_{t,r}^\varepsilon(x))| dr \right\}. \quad (3.10)$$

By Theorem 3.1, one has that for any $q > \frac{p\alpha}{p\alpha-d}$ and all $-1 \leq t \leq s_1 \leq s_2 \leq 0$,

$$\begin{aligned} \mathbb{E} \left(\int_{s_1}^{s_2} |\nabla u_r^\varepsilon(X_{t,r}^\varepsilon(x))| dr \Big| \mathcal{F}_{t,s_1} \right) &\leq C_{\|u^\varepsilon\|_\infty} \|\nabla u^\varepsilon\|_{L^q([s_1, s_2]; L^p)} \\ &\leq C_{\|u\|_{L^\infty([t, 0]; \mathbb{W}^{1,p})}} \|u\|_{L^q([s_1, s_2]; \mathbb{W}^{1,p})} \\ &\leq C_U U^q |s_2 - s_1|^{1/q}, \end{aligned}$$

where the second inequality is due to the Sobolev embedding relation $\mathbb{W}^{1,p} \subset L^\infty$ provided that $p > d$. Hence, by Lemma 3.2, for any $\gamma \geq 1$,

$$\sup_{\varepsilon \in (0,1)} \sup_{x \in \mathbb{R}^d} \mathbb{E} \exp \left\{ \gamma \int_t^0 |\nabla u_r^\varepsilon(X_{t,r}^\varepsilon(x))| dr \right\} < +\infty, \quad (3.11)$$

and one can choose $T \in [-1, 0)$ depending on U such that for all $t \in [T, 0)$,

$$\sup_{\varepsilon \in (0,1)} \sup_{x \in \mathbb{R}^d} \mathbb{E} \exp \left\{ 4 \int_t^0 |\nabla u_r^\varepsilon(X_{t,r}^\varepsilon(x))| dr \right\} \leq 2. \quad (3.12)$$

Let $A_{t,s}(x)$ solve the following linear random ODE:

$$A_{t,s}(x) = \mathbb{I} + \int_t^s \nabla u_r(X_{t,r}(x)) \cdot A_{t,r}(x) dr.$$

(Claim): For any $\delta \in [1, 2)$,

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \leq s; t, s \in [T, 0]} \sup_{x \in \mathbb{R}^d} \mathbb{E} |X_{t,s}^\varepsilon(x) - X_{t,s}(x)|^\delta = 0, \quad (3.13)$$

and for each $t \in [T, 0)$ and $x \in \mathbb{R}^d$,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} |\nabla X_{t,0}^\varepsilon(x) - A_{t,0}(x)|^\delta = 0. \quad (3.14)$$

We first prove (3.13). By (3.3), we have

$$\begin{aligned} |X_{t,s}^\varepsilon(x) - X_{t,s}(x)| &\leq \int_t^s |u_r^\varepsilon(X_{t,r}^\varepsilon(x)) - u_r^\varepsilon(X_{t,r}(x))| dr + \int_t^s |u_r^\varepsilon(X_{t,r}(x)) - u_r(X_{t,r}(x))| dr \\ &\leq C_d \int_t^s (\mathcal{M}|\nabla u_r^\varepsilon|(X_{t,r}^\varepsilon(x)) + \mathcal{M}|\nabla u_r^\varepsilon|(X_{t,r}(x))) |X_{t,r}^\varepsilon(x) - X_{t,r}(x)| dr \\ &\quad + \int_t^0 |u_r^\varepsilon(X_{t,r}(x)) - u_r(X_{t,r}(x))| dr, \end{aligned}$$

which yields by Gronwall's inequality that

$$\begin{aligned} |X_{t,s}^\varepsilon(x) - X_{t,s}(x)| &\leq \exp \left\{ C_d \int_t^s (\mathcal{M}|\nabla u_r^\varepsilon|(X_{t,r}^\varepsilon(x)) + \mathcal{M}|\nabla u_r^\varepsilon|(\hat{X}_{t,r}(x))) dr \right\} \\ &\quad \times \int_t^0 |u_r^\varepsilon(X_{t,r}(x)) - u_r(X_{t,r}(x))| dr. \end{aligned}$$

As in estimating (3.11), one has that for any $\gamma \geq 1$,

$$\sup_{\varepsilon \in (0,1)} \sup_{x \in \mathbb{R}^d} \mathbb{E} \exp \left\{ \gamma \int_t^0 \mathcal{M}|\nabla u_r^\varepsilon|(X_{t,r}^\varepsilon(x)) dr \right\} < +\infty$$

and

$$\sup_{\varepsilon \in (0,1)} \sup_{x \in \mathbb{R}^d} \mathbb{E} \exp \left\{ \gamma \int_t^0 \mathcal{M}|\nabla u_r^\varepsilon|(X_{t,r}(x)) dr \right\} < +\infty.$$

Hence, by Hölder's inequality and Theorem 3.1 again, we have for any $\delta \in [1, 2)$ and $q > \frac{p\alpha}{p\alpha - 2d}$,

$$\begin{aligned} \mathbb{E} |X_{t,s}^\varepsilon(x) - X_{t,s}(x)|^\delta &\leq \left(\mathbb{E} \exp \left\{ \frac{4\delta C_d}{2-\delta} \int_t^0 \mathcal{M}|\nabla u_r^\varepsilon|(X_{t,r}^\varepsilon(x)) dr \right\} \right)^{(2-\delta)/4} \\ &\quad \times \left(\mathbb{E} \exp \left\{ \frac{4\delta C_d}{2-\delta} \int_t^0 \mathcal{M}|\nabla u_r^\varepsilon|(X_{t,r}(x)) dr \right\} \right)^{(2-\delta)/4} \\ &\quad \times \left(|t| \mathbb{E} \int_t^0 |u_r^\varepsilon - u_r|^2(X_{t,r}(x)) dr \right)^{\delta/2} \\ &\leq C \| |u^\varepsilon - u|^2 \|_{L^q([t,0]; L^{p/2})}^{\delta/2} \\ &= C \| u^\varepsilon - u \|_{L^{2q}([t,0]; L^p)}^\delta, \end{aligned} \quad (3.15)$$

where C is independent of ε, x, t, s . Limit (3.13) now follows from the property of convolutions.

As for (3.14), we have

$$\begin{aligned} |\nabla X_{t,s}^\varepsilon(x) - A_{t,s}(x)| &\leq \int_t^s |\nabla u_r^\varepsilon(X_{t,r}^\varepsilon(x)) - \nabla u_r(X_{t,r}(x))| \cdot |\nabla X_{t,r}^\varepsilon(x)| dr \\ &\quad + \int_t^s |\nabla u_r(X_{t,r}(x))| \cdot |\nabla X_{t,r}^\varepsilon(x) - A_{t,r}(x)| dr, \end{aligned}$$

which then gives that

$$|\nabla X_{t,0}^\varepsilon(x) - A_{t,0}(x)| \leq \int_t^0 |\nabla u_r^\varepsilon(X_{t,r}^\varepsilon(x)) - \nabla u_r(X_{t,r}(x))| \cdot |\nabla X_{t,r}^\varepsilon(x)| dr$$

$$\times \exp \left\{ \int_t^0 |\nabla u_r(X_{t,r}(x))| dr \right\}.$$

As above, using (3.10), (3.11) and Hölder's inequality, we have

$$\mathbb{E} |\nabla X_{t,0}^\varepsilon(x) - A_{t,0}(x)|^\delta \leq C \left(\mathbb{E} \int_t^0 |\nabla u_r^\varepsilon(X_{t,r}^\varepsilon(x)) - \nabla u_r(X_{t,r}(x))|^2 dr \right)^{\delta/2}. \quad (3.16)$$

For fixed $\varepsilon' \in (0, 1)$, by (3.13), we have

$$\mathbb{E} \int_t^0 |\nabla u_r^{\varepsilon'}(X_{t,r}^\varepsilon(x)) - \nabla u_r^{\varepsilon'}(X_{t,r}(x))|^2 dr \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \quad (3.17)$$

By (3.2), we have for $q > \frac{p\alpha}{p\alpha-2d}$,

$$\sup_{\varepsilon \in (0,1)} \mathbb{E} \int_t^0 |\nabla(u_r^{\varepsilon'} - u_r)(X_{t,r}^\varepsilon(x))|^2 dr \leq C \|\nabla(u^{\varepsilon'} - u)\|_{L^{2q}([t,0];L^p)}^2 \rightarrow 0 \text{ as } \varepsilon' \rightarrow 0. \quad (3.18)$$

Limit (3.14) then follows by (3.16), (3.17) and (3.18).

Using the above claims, by (3.10), (3.11) and (3.12), one finds that $X_{t,s}(\cdot) \in \cap_{\gamma \geq 1} \mathbb{W}_{loc}^{1,\gamma}$, and (3.6) and (3.7) hold. Moreover, if we define

$$w_t^\varepsilon := \mathbf{P} \mathbb{E}(\nabla^t X_{t,0}^\varepsilon \cdot (\varphi \circ X_{t,0}^\varepsilon)),$$

then, since \mathbf{P} is a bounded linear operator in L^p and $x \mapsto X_{t,0}^\varepsilon(x)$ preserves the volume, it is easy to see that $w^\varepsilon \in C([T, 0]; \cap_{k \in \mathbb{N}} \mathbb{W}^{k,p})$. Noting that

$$0 = \mathbf{P} \nabla \mathbb{E}(X_{t,0}^\varepsilon \cdot (\varphi \circ X_{t,0}^\varepsilon)) = \mathbf{P} \mathbb{E}(\nabla^t X_{t,0}^\varepsilon \cdot (\varphi \circ X_{t,0}^\varepsilon)) + \mathbf{P} \mathbb{E}(\nabla^t(\varphi \circ X_{t,0}^\varepsilon) \cdot X_{t,0}^\varepsilon), \quad (3.19)$$

we have

$$\partial_t w_t^\varepsilon = \mathbf{P} \mathbb{E}[\nabla^t X_{t,0}^\varepsilon \cdot (\nabla \varphi - \nabla^t \varphi) \circ X_{t,0}^\varepsilon \cdot \partial_t X_{t,0}^\varepsilon].$$

Using limits (3.13) and (3.14), formula (3.9) then follows, and meanwhile,

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [T,0]} \|w_t^\varepsilon - w_t\|_{1,p} = 0.$$

So, $w \in C([T, 0]; \mathbb{W}^{1,p})$. The proof is complete. \square

The following lemma gives the continuous dependence of solutions to SDE (3.1) with respect to u .

Lemma 3.5. *Suppose that $(\mathbf{H})_\alpha$ holds with $\alpha \in (1, 2)$, and $p > \frac{2d}{\alpha}$. For $U > 0$ and $T \in [-1, 0)$, let $u, \hat{u} \in L^\infty([T, 0]; \mathbb{W}^{1,p}(\mathbb{R}^d; \mathbb{R}^d))$ be divergence free with*

$$\sup_{t \in [T,0]} \|u_t\|_{1,p} \leq U, \quad \sup_{t \in [T,0]} \|\hat{u}_t\|_{1,p} \leq U.$$

Let X, \hat{X} be the solutions of SDE (3.1) corresponding to u, \hat{u} . Then for any $\delta \in [1, 2)$, $q > \frac{p\alpha}{p\alpha-2d}$ and $t \in [T, 0]$,

$$\sup_{x \in \mathbb{R}^d} \mathbb{E} |X_{t,0}(x) - \hat{X}_{t,0}(x)|^\delta \leq C_1 \|u - \hat{u}\|_{L^{2q}([t,0];L^p)}^\delta, \quad (3.20)$$

where C_1 only depends on $U, T, \alpha, \delta, p, q, d, \psi$. Moreover, for any $\varphi, \hat{\varphi} \in \mathbb{W}^{1,p}(\mathbb{R}^d; \mathbb{R}^d)$, let w_t and \hat{w}_t be defined as in (3.8) corresponding to (φ, X) and $(\hat{\varphi}, \hat{X})$, then for any $q > \frac{p\alpha}{p\alpha-2d}$ and $t \in [T, 0]$,

$$\|w_t - \hat{w}_t\|_p^{2q} \leq C_2 \|\varphi - \hat{\varphi}\|_p^{2q} + C_3 \int_t^0 \|u_r - \hat{u}_r\|_p^{2q} dr, \quad (3.21)$$

where C_2 (resp. C_3) only depends on $U, T, \alpha, p, q, d, \psi$ (resp. $U, T, \alpha, p, q, d, \psi, \|\varphi\|_{1,p}$).

Proof. Estimate (3.20) follows from the same calculations as in estimating (3.15). Let us look at (3.21). Using mollifying approximation and (3.19), we have

$$\begin{aligned} w_t - \hat{w}_t &= \mathbf{P}\mathbb{E}[\nabla^t(X_{t,0} - \hat{X}_{t,0}) \cdot (\varphi \circ X_{t,0})] \\ &\quad + \mathbf{P}\mathbb{E}[\nabla^t \hat{X}_{t,0} \cdot (\varphi \circ X_{t,0} - \hat{\varphi} \circ \hat{X}_{t,0})] \\ &= \mathbf{P}\mathbb{E}[\nabla^t(\varphi \circ X_{t,0}) \cdot (X_{t,0} - \hat{X}_{t,0})] \\ &\quad + \mathbf{P}\mathbb{E}[\nabla^t \hat{X}_{t,0} \cdot (\varphi \circ X_{t,0} - \varphi \circ \hat{X}_{t,0})] \\ &\quad + \mathbf{P}\mathbb{E}[\nabla^t \hat{X}_{t,0} \cdot (\varphi \circ \hat{X}_{t,0} - \hat{\varphi} \circ \hat{X}_{t,0})]. \end{aligned}$$

By the boundedness of \mathbf{P} in L^p , we have

$$\begin{aligned} \|w_t - \hat{w}_t\|_p &\leq C\|\mathbb{E}[\nabla^t(\varphi \circ X_{t,0}) \cdot (X_{t,0} - \hat{X}_{t,0})]\|_p \\ &\quad + C\|\mathbb{E}[\nabla^t \hat{X}_{t,0} \cdot (\varphi \circ X_{t,0} - \varphi \circ \hat{X}_{t,0})]\|_p \\ &\quad + C\|\mathbb{E}[\nabla^t \hat{X}_{t,0} \cdot (\varphi \circ \hat{X}_{t,0} - \hat{\varphi} \circ \hat{X}_{t,0})]\|_p. \end{aligned}$$

By Hölder's inequality, for any $\delta \in (p/(p-1), 2)$ and some $\beta = \beta(\delta, p) > 1$, we have

$$\begin{aligned} &\|\mathbb{E}[\nabla^t \hat{X}_{t,0} \cdot (\varphi \circ X_{t,0} - \varphi \circ \hat{X}_{t,0})]\|_p \\ &\stackrel{(3.3)}{\leq} C\|\mathbb{E}[|\nabla \hat{X}_{t,0}| \cdot (\mathcal{M}|\nabla \varphi|(X_{t,0}) + \mathcal{M}|\nabla \varphi|(\hat{X}_{t,0})) \cdot |X_{t,0} - \hat{X}_{t,0}|]\|_p \\ &\leq C\|\nabla \hat{X}_{t,0}\|_{L^\beta(\Omega)} \cdot \|\mathcal{M}|\nabla \varphi|(X_{t,0}) + \mathcal{M}|\nabla \varphi|(\hat{X}_{t,0})\|_{L^p(\Omega)} \cdot \|X_{t,0} - \hat{X}_{t,0}\|_{L^\delta(\Omega)} \\ &\leq C \sup_{x \in \mathbb{R}^d} \|\nabla \hat{X}_{t,0}(x)\|_{L^\beta(\Omega)} \cdot \sup_{x \in \mathbb{R}^d} \|X_{t,0}(x) - \hat{X}_{t,0}(x)\|_{L^\delta(\Omega)} \cdot \|\mathcal{M}|\nabla \varphi|\|_p \\ &\stackrel{(3.6)(3.20)(3.4)}{\leq} C\|u - \hat{u}\|_{L^{2q}([t,0];L^p)} \cdot \|\nabla \varphi\|_p, \end{aligned}$$

where we have used that $x \mapsto X_{t,0}(x), \hat{X}_{t,0}(x)$ preserve the volume. Similarly, we have

$$\|\mathbb{E}[\nabla^t(\varphi \circ X_{t,0}) \cdot (X_{t,0} - \hat{X}_{t,0})]\|_p \leq C\|u - \hat{u}\|_{L^{2q}([t,0];L^p)} \cdot \|\nabla \varphi\|_p,$$

and

$$\|\mathbb{E}[\nabla^t \hat{X}_{t,0} \cdot (\varphi \circ \hat{X}_{t,0} - \hat{\varphi} \circ \hat{X}_{t,0})]\|_p \leq C\|\varphi - \hat{\varphi}\|_p.$$

The proof is thus complete. \square

We are now in a position to prove the following main result of this section.

Theorem 3.6. *For any divergence free $u_0 \in \mathbb{W}^{1,p}(\mathbb{R}^d; \mathbb{R}^d)$ with $p > \frac{2d}{\alpha}$, there exist a time $T = T(\|u_0\|_{1,p}) < \infty$ independent of $\nu \geq 1$ and a unique pair of (u, X) with $u \in C([T, 0], \mathbb{W}^{1,p})$ solving the following stochastic system:*

$$\begin{cases} X_{t,s}(x) = x + \int_t^s u_r(X_{t,r}(x))dr + \nu^{1/\alpha}(L_s - L_t), t \leq s \leq 0, \\ u_t = \mathbf{P}\mathbb{E}[\nabla^t X_{t,0} \cdot (u_0 \circ X_{t,0})]. \end{cases} \quad (3.22)$$

Moreover, for some $C_0 \geq 1$ only depending on p ,

$$\sup_{t \in [T, 0]} \|u_t\|_{1,p} \leq 3C_0\|u_0\|_{1,p},$$

and u satisfies (1.5) in a generalized sense, i.e., for all divergence free vector field $\phi \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d)$,

$$\langle u_t, \phi \rangle = \langle u_0, \phi \rangle + \int_t^0 [\langle u_s, \mathcal{L}_\nu^* \phi \rangle + \langle (u_s \cdot \nabla) u_s, \phi \rangle] ds, \quad (3.23)$$

where

$$\mathcal{L}_v^* g(x) = \int_{\mathbb{R}^d} \left[g(x - v^{1/\alpha} y) - g(x) + (v^{1/\alpha} y \cdot \nabla) g(x) 1_{|y| \leq 1} \right] \nu(dy).$$

Proof. Set $u_r^0(x) \equiv u_0(x)$. For $n \in \mathbb{N}_0$, by Lemma 3.4, we recursively define

$$\begin{cases} X_{t,s}^n(x) = x + \int_t^s u_r^n(X_{t,r}^n(x)) dr + v^{1/\alpha} (L_s - L_t), \\ u_t^{n+1} = \mathbf{P} \mathbb{E}[\nabla^t X_{t,0}^n \cdot (u_0 \circ X_{t,0}^n)]. \end{cases} \quad (3.24)$$

Let us estimate the $\mathbb{W}^{1,p}$ -norm of u^{n+1} . First of all, by Hölder's inequality, we have

$$\begin{aligned} \|u_t^{n+1}\|_p &\leq C \|\mathbb{E}[\nabla^t X_{t,0}^n \cdot (u_0 \circ X_{t,0}^n)]\|_p \\ &\leq C \left(\int_{\mathbb{R}^d} \|\nabla X_{t,0}^n(x)\|_{L^2(\Omega)}^p \cdot \|u_0 \circ X_{t,0}^n(x)\|_{L^2(\Omega)}^p dx \right)^{1/p} \\ &\leq C \sup_{x \in \mathbb{R}^d} \|\nabla X_{t,0}^n(x)\|_{L^2(\Omega)} \left(\int_{\mathbb{R}^d} \mathbb{E}|u_0 \circ X_{t,0}^n(x)|^p dx \right)^{1/p} \\ &= C \sup_{x \in \mathbb{R}^d} \|\nabla X_{t,0}^n(x)\|_{L^2(\Omega)} \|u_0\|_p, \end{aligned}$$

and by (3.9),

$$\|\nabla u_t^{n+1}\|_p \leq C \|\mathbb{E}[|\nabla X_{t,0}^n|^2 \cdot |\nabla u_0 \circ X_{t,0}^n|]\|_p \leq C \sup_{x \in \mathbb{R}^d} \|\nabla X_{t,0}^n(x)\|_{L^4(\Omega)}^2 \|\nabla u_0\|_p,$$

Hence, for some $C_0 \geq 1$ only depending on p ,

$$\|u_t^{n+1}\|_{1,p} \leq C_0 \left(1 + \sup_{x \in \mathbb{R}^d} \|\nabla X_{t,0}^n(x)\|_{L^4(\Omega)}^4 \right) \|u_0\|_{1,p}.$$

Now, taking $U = 3C_0 \|u_0\|_{1,p}$ in Lemma 3.4, by induction method, there exists a time $T = T(U) < 0$ independent of $v \geq 1$ such that for all $n \in \mathbb{N}_0$,

$$\sup_{t \in [T, 0]} \|u_t^{n+1}\|_{1,p} \leq U, \quad \sup_{t \in [T, 0]} \sup_{x \in \mathbb{R}^d} \mathbb{E}|\nabla X_{t,0}^n(x)|^4 \leq 2, \quad (3.25)$$

and for any $\gamma \geq 1$,

$$\sup_{n \in \mathbb{N}_0} \sup_{t \in [T, 0]} \sup_{x \in \mathbb{R}^d} \mathbb{E}|\nabla X_{t,0}^n(x)|^\gamma < +\infty.$$

Thus, by Lemma 3.5, one has that for all $t \in [T, 0]$,

$$\|u_t^{n+1} - u_t^{m+1}\|_p^{2q} \leq C \int_t^0 \|u_r^n - u_r^m\|_p^{2q} dr,$$

where the constant C is independent of n, m and t . By Gronwall's inequality, we get

$$\lim_{n, m \rightarrow \infty} \sup_{t \in [T, 0]} \|u_t^n - u_t^m\|_p^{2q} = 0.$$

Thus, by (3.25), there exists a $u \in L^\infty([T, 0]; \mathbb{W}^{1,p})$ such that

$$\lim_{n \rightarrow \infty} \sup_{t \in [T, 0]} \|u_t^n - u_t\|_p = 0$$

and

$$\sup_{t \in [T, 0]} \|u_t\|_{1,p} \leq U.$$

Let X be the solution of SDE (3.1) corresponding to the above u . By taking limits for both sides of (3.24) and using Lemma 3.5, one finds that (u, X) solves (3.22). Moreover, the regularity of $u \in C([T, 0]; \mathbb{W}^{1,p})$ follows from Lemma 3.4, and the uniqueness follows from Lemma 3.5. As for equation (3.23), let $u_\varepsilon^0(x) := u_0 * \rho_\varepsilon(x)$ be the mollifying approximation of u_0 . Let

$u_t^\varepsilon(x) \in C([T, 0]; \cap_{k \geq 0} \mathbb{W}^{k,p})$ be the corresponding solution of (3.22). By Theorem 2.3, one has that for all divergence free vector field $\phi \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d)$

$$\langle u_t^\varepsilon, \phi \rangle = \langle u_0^\varepsilon, \phi \rangle + \int_t^0 [\langle u_s^\varepsilon, \mathcal{L}_v^* \phi \rangle + \langle (u_s^\varepsilon \cdot \nabla) u_s^\varepsilon, \phi \rangle] ds.$$

Taking limits $\varepsilon \rightarrow 0$ and by Lemma 3.5, one then obtains equation (3.23). \square

4. GRADIENT ESTIMATES FOR LÉVY PROCESSES WITH DRIFTS

In this section we prove gradient estimates for SDE (3.1). It will be used to obtain the global well posedness for stochastic system (3.22) with large viscosity ν in the periodic case. This section can be read independently and has some interests in itself.

Let $\mathcal{P}(\mathbb{R}^d)$ be the space of all Borel probability measures on \mathbb{R}^d . For $\mu \in \mathcal{P}(\mathbb{R}^d)$, define

$$\mathcal{T}_\mu f(x) = \int_{\mathbb{R}^d} f(x+y) \mu(dy), \quad f \in C_b(\mathbb{R}^d).$$

It is clear that for any $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$,

$$\mathcal{T}_\mu \mathcal{T}_\nu f = \mathcal{T}_{\mu * \nu} f, \quad f \in C_b(\mathbb{R}^d),$$

where $\mu * \nu$ denotes the convolution between μ and ν .

For $a > 0$, let us consider the truncated symbol of $\psi(\xi)$ in (2.1):

$$\psi_a(\xi) = \int_{0 < |x| \leq a} (1 - e^{i\xi \cdot x} + i\xi \cdot x 1_{|x| \leq 1}) \nu(dx).$$

For any $t < 0$, by Bochner's theorem, there exists a unique probability measure $\mu_{a,t} \in \mathcal{P}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} e^{i\xi \cdot x} \mu_{a,t}(dx) = e^{t\psi_a(\xi)}, \quad (4.1)$$

and respectively, $\tilde{\mu}_{a,t} \in \mathcal{P}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} e^{i\xi \cdot x} \tilde{\mu}_{a,t}(dx) = e^{t(\psi(\xi) - \psi_a(\xi))}. \quad (4.2)$$

We recall the following result from [20, Proposition 2.3].

Proposition 4.1. *Assume that $(\mathbf{H})_\alpha$ holds with $\alpha \in (0, 2)$. Then for any $a, t > 0$,*

$$\mu_{a,t}(dx) = p_{a,t}(x) dx,$$

and for any $n \in \mathbb{N}_0$, there exists a time $t_0 = t_0(n, d, \alpha, \psi) < 0$ and a constant $C = C(t_0, n, d, \alpha, \psi) > 0$ such that for any $t \in [t_0, 0)$,

$$|\nabla^n p_{|t|^{1/\alpha}, t}(x)| \leq C |t|^{-(d+n)/\alpha} (1 + |t|^{-1/\alpha} |x|)^{-d-1}. \quad (4.3)$$

For $\beta > 0$, let $\mathbb{W}^{\beta,p} := (I - \Delta)^{-\beta/2}(L^p)$ be the Bessel potential space with the norm:

$$\|f\|_{\beta,p} := \|(I - \Delta)^{\beta/2} f\|_p.$$

If $\beta = k \in \mathbb{N}$, $\mathbb{W}^{\beta,p}$ is the same as the Sobolev space $\mathbb{W}^{k,p}$. Using Proposition 4.1, we now derive the following useful result.

Theorem 4.2. *Assume that $(\mathbf{H})_\alpha$ holds with $\alpha \in (0, 2)$. For $\nu > 0$, define*

$$\mathcal{T}_t^\nu f(x) := \mathbb{E}(f(\nu^{\frac{1}{\alpha}} L_t + x)). \quad (4.4)$$

Then for any $p \in [1, \infty]$ and $m, n \in \mathbb{N}$, there exists a constant $C = C(p, m, n, d, \psi)$ such that for all $f \in \mathbb{W}^{m,p}$,

$$\|\mathcal{T}_t^\nu f\|_{m+n,p} \leq C[\nu(|t| \wedge 1)]^{-m/\alpha} \|f\|_{n,p}, \quad \forall \nu > 0, \forall t < 0. \quad (4.5)$$

If $p \in (1, \infty)$, the above estimate also holds for any nonnegative real numbers m, n .

Proof. For $\nu, a > 0$, let $\mu_{a,t}^\nu, \tilde{\mu}_{a,t}^\nu \in \mathcal{P}(\mathbb{R}^d)$ be defined as in (4.1) and (4.2) corresponding to the symbols $\psi_a(\nu^{1/\alpha}\xi)$ and $\psi(\nu^{1/\alpha}\xi) - \psi_a(\nu^{1/\alpha}\xi)$. It is easy to see that

$$\mu_{a,t}^\nu(dx) = \nu^{-\frac{d}{\alpha}} p_{a,t}(\nu^{-\frac{1}{\alpha}}x)dx. \quad (4.6)$$

Set

$$\mathcal{T}_{a,t}^\nu f = \mathcal{T}_{\mu_{a,t}^\nu} f, \quad \tilde{\mathcal{T}}_{a,t}^\nu f = \mathcal{T}_{\tilde{\mu}_{a,t}^\nu} f,$$

then for all $t \in \mathbb{R}_-$,

$$\mathcal{T}_t^\nu f = \mathcal{T}_{a,t}^\nu \tilde{\mathcal{T}}_{a,t}^\nu f. \quad (4.7)$$

By (4.6) and Proposition 4.1, there exists a time $t_0 = t_0(n, d, \alpha, \psi) < 0$ such that for all $t \in [t_0, 0)$ and $f \in L^p(\mathbb{R}^d)$,

$$\begin{aligned} \nabla^n \mathcal{T}_{|t|^{1/\alpha}, t}^\nu f(x) &= \nu^{-\frac{d+n}{\alpha}} \int_{\mathbb{R}^d} f(y) \nabla^n p_{|t|^{1/\alpha}, t}(\nu^{-\frac{1}{\alpha}}(y-x)) dy \\ &= \nu^{-\frac{d+n}{\alpha}} \int_{\mathbb{R}^d} f(x+y) \nabla^n p_{|t|^{1/\alpha}, t}(\nu^{-\frac{1}{\alpha}}y) dy. \end{aligned}$$

Hence, for any $p \in [1, \infty]$, by Minkowskii's inequality and (4.3), we have

$$\begin{aligned} \|\nabla^n \mathcal{T}_{|t|^{1/\alpha}, t}^\nu f\|_p &\leq \nu^{-\frac{d+n}{\alpha}} \|f\|_p \int_{\mathbb{R}^d} |\nabla^n p_{|t|^{1/\alpha}, t}(\nu^{-\frac{1}{\alpha}}y)| dy \\ &\leq C(\nu|t|)^{-(d+n)/\alpha} \|f\|_p \int_{\mathbb{R}^d} (1 + (\nu|t|)^{-1/\alpha}|y|)^{-d-1} dy \\ &= C(\nu|t|)^{-n/\alpha} \|f\|_p \int_{\mathbb{R}^d} (1 + |y|)^{-d-1} dy = \tilde{C}(\nu|t|)^{-n/\alpha} \|f\|_p. \end{aligned} \quad (4.8)$$

Thus, by (4.7) and the L^p -contraction of $\tilde{\mathcal{T}}_{a,t}^\nu$, we have

$$\|\nabla^{n+m} \mathcal{T}_t^\nu f\|_p \leq \tilde{C}(\nu|t|)^{-n/\alpha} \|\tilde{\mathcal{T}}_{|t|^{1/\alpha}, t}^\nu \nabla^m f\|_p \leq \tilde{C}(\nu|t|)^{-n/\alpha} \|\nabla^m f\|_p,$$

which yields that

$$\|\mathcal{T}_t^\nu f\|_{m+n,p} \leq \tilde{C}(\nu|t|)^{-n/\alpha} \|f\|_{m,p}.$$

For general $t < t_0$, it follows by the semigroup property of \mathcal{T}_t^ν . If $p \in (1, \infty)$ and m, n are nonnegative real numbers, it follows by interpolation theorem (cf. [23]). \square

Let us consider the following PIDE:

$$\partial_t h + \mathcal{L}_\nu h + (u \cdot \nabla)h = 0, \quad t \leq 0, \quad (4.9)$$

subject to the final value

$$h_0(x) = \varphi(x),$$

where

$$\mathcal{L}_\nu u(x) := \int_{\mathbb{R}^d} [u(x + \nu^{1/\alpha}y) - u(x) - (\nu^{1/\alpha}y \cdot \nabla)u(x) 1_{|y| \leq 1}] \nu(dy).$$

By Duhamel's formula, one can write equation (4.9) as the following mild form:

$$h_t(x) = \mathcal{T}_t^\nu \varphi(x) + \int_t^0 \mathcal{T}_{t-s}^\nu ((u_s \cdot \nabla)h_s)(x) ds, \quad (4.10)$$

where \mathcal{T}_t^ν is defined by (4.4).

We need the following simple lemma (cf. [26, Lemma 5.1]).

Lemma 4.3. *Let $z(t)$ be a nonnegative function defined on $[-1, 0)$, and satisfy that for some $K_1, K_2 > 0$ and $\beta, \gamma \in (0, 1)$,*

$$z(t) \leq K_1 |t|^{-\gamma} + K_2 \int_t^0 \frac{z(s)}{(s-t)^\beta} ds, \quad \forall t \in [-1, 0).$$

Then for any $t \in [-1, 0)$

$$z(t) \leq C_{K_2, \beta, \gamma} K_1 |t|^{-\gamma}.$$

We have the following useful estimate.

Lemma 4.4. *Assume that $(\mathbf{H})_\alpha$ holds with $\alpha \in (1, 2)$ and*

$$u \in L^\infty([-1, 0]; L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d))$$

provided that $p > \frac{d}{\alpha-1}$. Then for any $\gamma \in (\frac{d}{p} + 1, \alpha)$ and $\varphi \in L^p(\mathbb{R}^d)$, there exists a unique solution $h \in L^1([-1, 0]; \mathbb{W}^{\gamma, p})$ to equation (4.10) such that for all $t \in [-1, 0)$ and $v \geq 1$,

$$\|h_t\|_{\gamma, p} \leq C_1 (v|t|)^{-\gamma/\alpha} \|\varphi\|_p, \quad (4.11)$$

where C_1 only depends on $\gamma, p, d, \alpha, \psi$ and the norm of $\|u\|_{L^\infty([-1, 0]; L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d))}$. In the case of $u \in L^\infty([-1, 0] \times \mathbb{R}^d)$, for any $p \in [1, \infty]$ and $\varphi \in L^p(\mathbb{R}^d)$, there exists a unique solution $h \in L^1([-1, 0]; \mathbb{W}^{1, p})$ to equation (4.10) such that for all $t \in [-1, 0)$ and $v \geq 1$,

$$\|\nabla h_t\|_p \leq C_2 (v|t|)^{-1/\alpha} \|\varphi\|_p, \quad (4.12)$$

where C_2 only depends on p, d, α, ψ and the bound of u , and is increasing with respect to the bound of u .

Proof. We only prove the a priori estimates (4.11) and (4.12). As for the existence, it follows from a standard Picardi iteration argument. Assume that $u = u_1 + u_2$ with $u_1 \in L^\infty([-1, 0]; L^p(\mathbb{R}^d))$ and $u_2 \in L^\infty([-1, 0]; L^\infty(\mathbb{R}^d))$. Then, by (4.5) and the Sobolev embedding $\mathbb{W}^{\gamma, p} \subset C_b^1$ provided $(\gamma - 1)p > d$, we have

$$\begin{aligned} \|h_t\|_{\gamma, p} &\leq \|\mathcal{T}_t^\gamma \varphi\|_{\gamma, p} + C \int_t^0 (v(s-t))^{-\frac{\gamma}{\alpha}} \|(u_s \cdot \nabla) h_s\|_p ds \\ &\leq \|\mathcal{T}_t^\gamma \varphi\|_{\gamma, p} + C \int_t^0 (s-t)^{-\frac{\gamma}{\alpha}} (\|u_{1,s}\|_\infty \cdot \|\nabla h_s\|_p + \|u_{2,s}\|_p \cdot \|\nabla h_s\|_\infty) ds \\ &\leq C(v|t|)^{-\gamma/\alpha} \|\varphi\|_p + C \int_t^0 (s-t)^{-\frac{\gamma}{\alpha}} (\|u_{1,s}\|_\infty + \|u_{2,s}\|_p) \cdot \|\nabla h_s\|_{\gamma, p} ds. \end{aligned}$$

By Lemma 4.3, we obtain (4.11).

In the case of $u \in L^1([-1, 0]; L^\infty(\mathbb{R}^d))$, one has

$$\begin{aligned} \|\nabla h_t\|_p &\leq \|\nabla \mathcal{T}_t^\gamma \varphi\|_p + C \int_t^0 (v(s-t))^{-\frac{1}{\alpha}} \|(u_s \cdot \nabla) h_s\|_p ds \\ &\leq C(v|t|)^{-1/\alpha} \|\varphi\|_p + C \int_t^0 (s-t)^{-\frac{1}{\alpha}} \|u_s\|_\infty \cdot \|\nabla h_s\|_p ds, \end{aligned}$$

which gives (4.12) by Lemma 4.3 again. \square

Before proving the gradient estimates for SDE (3.1), we recall the notions of weak existence and uniqueness for SDE (3.1). Fix $t < 0$. Let \mathbb{D}_t be the space of all càdlàg functions from $[t, 0] \rightarrow \mathbb{R}^d$. We endow \mathbb{D}_t with the Skorohod metric so that \mathbb{D}_t is a Polish space. A weak solution of SDE (3.1) means that there exists a probability space (Ω, \mathcal{F}, P) and two càdlàg

processes $(X_{t,s})_{s \in [t,0]}$ and $(L_{t,s})_{s \in [t,0]}$ defined on it such that $s \mapsto L_{t,s}$ is a Lévy process with symbol $\psi(\xi)$ and

$$X_{t,s}(x) = x + \int_t^s u_r(X_{t,r}(x)) dr + \nu^{1/\alpha} L_{t,s}, \quad s \in [t, 0], \quad P - a.s.$$

Such a solution will be denoted by $(\Omega, \mathcal{F}, P; (X_{t,s})_{s \in [t,0]}, (L_{t,s})_{s \in [t,0]})$. Weak uniqueness means that for two weak solutions $(\Omega^{(i)}, \mathcal{F}^{(i)}, P^{(i)}; (X_{t,s}^{(i)})_{s \in [t,0]}, (L_{t,s}^{(i)})_{s \in [t,0]})$, $i = 1, 2$, the laws of $s \mapsto X_{t,s}^{(i)}$ in \mathbb{D}_t are the same for $i = 1, 2$.

Assume that for each $x \in \mathbb{R}^d$, weak uniqueness holds for SDE (3.1). Then for any bounded measurable function φ , the mapping

$$x \mapsto \mathbb{E}\varphi(X_{t,0}(x))$$

is well defined. Now, we can prove the following gradient estimate for $x \mapsto \mathbb{E}\varphi(X_{t,0}(x))$.

Theorem 4.5. Assume that $(\mathbf{H})_\alpha$ holds with $\alpha \in (1, 2)$ and

$$u \in L^\infty([-1, 0]; L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)) \quad (4.13)$$

provided that $p > \frac{d}{\alpha-1}$. We also assume that for each $x \in \mathbb{R}^d$, weak uniqueness holds for SDE (3.1). Then for any $\gamma \in (\frac{d}{p} + 1, \alpha)$, $\nu \geq 1$ and $\varphi \in L^p(\mathbb{R}^d)$,

$$\|\mathbb{E}\varphi(X_{t,0}(\cdot))\|_{\gamma,p} \leq C_1(\nu|t|)^{-\gamma/\alpha} \|\varphi\|_p; \quad (4.14)$$

in the case of $u \in L^\infty([-1, 0] \times \mathbb{R}^d)$, for any $p \in [1, \infty]$, $\nu \geq 1$ and $\varphi \in L^p(\mathbb{R}^d)$,

$$\|\nabla \mathbb{E}\varphi(X_{t,0}(\cdot))\|_p \leq C_2(\nu|t|)^{-1/\alpha} \|\varphi\|_p, \quad (4.15)$$

where C_1 and C_2 are the same as in Lemma 4.4.

Proof. Let $u_t^\varepsilon(x) = u_t * \rho_\varepsilon(x)$ be the mollifying approximation of u_t , and $X_{t,s}^\varepsilon$ be the corresponding solution of SDE (3.1). For $\varphi \in C_b^\infty(\mathbb{R}^d)$, define

$$h_t^\varepsilon(x) := \mathbb{E}\varphi(X_{t,0}^\varepsilon(x)).$$

By Theorem 2.1, u^ε solves the following PIDE:

$$\partial_t h^\varepsilon + \mathcal{L}_\nu h^\varepsilon + (u^\varepsilon \cdot \nabla) h^\varepsilon = 0,$$

which is equivalent by Duhamel's formula that,

$$h_t^\varepsilon(x) = \mathcal{T}_t^\nu \varphi(x) + \int_t^0 \mathcal{T}_{t-s}^\nu ((u_s^\varepsilon \cdot \nabla) h_s^\varepsilon)(x) ds.$$

Under (4.13), it is well known that the laws of $\{X_{t,\cdot}^\varepsilon(x)\}_{\varepsilon \in (0,1)}$ in \mathbb{D}_t is tight and any accumulation point is a weak solution of SDE (3.1) (cf. [28, Theorem 4.1]). By the weak uniqueness, the whole sequence of $X_{t,\cdot}^\varepsilon(x)$ weakly converges to the weak solution $X_{t,\cdot}(x)$ in \mathbb{D}_t . In particular, for any $x \in \mathbb{R}^d$,

$$\lim_{\varepsilon \rightarrow 0} h_t^\varepsilon(x) = \lim_{\varepsilon \rightarrow 0} \mathbb{E}\varphi(X_{t,0}^\varepsilon(x)) = \mathbb{E}\varphi(X_{t,0}(x)). \quad (4.16)$$

By (4.11), we have for all $\varphi \in C_0^\infty(\mathbb{R}^d)$,

$$\|\mathbb{E}\varphi(X_{t,0}^\varepsilon(\cdot))\|_{\gamma,p} = \|h_t^\varepsilon\|_{\gamma,p} \leq C(\nu|t|)^{-\gamma/\alpha} \|\varphi\|_p.$$

By (4.16), we get (4.15) for any $\varphi \in C_0^\infty(\mathbb{R}^d)$. For general $\varphi \in L^p(\mathbb{R}^d)$, it follows by a standard approximation.

In the case of $u \in L^\infty([-1, 0] \times \mathbb{R}^d)$, by (4.12), for $p = \infty$, we have

$$|\mathbb{E}\varphi(X_{t,0}^\varepsilon(x)) - \mathbb{E}\varphi(X_{t,0}^\varepsilon(y))| = |h_t^\varepsilon(x) - h_t^\varepsilon(y)| \leq C(\nu|t|)^{-1/\alpha} \|\varphi\|_\infty |x - y|.$$

Letting $\varepsilon \rightarrow 0$ yields that

$$|\mathbb{E}\varphi(X_{t,0}(x)) - \mathbb{E}\varphi(X_{t,0}(y))| \leq C(v|t|)^{-1/\alpha} \|\varphi\|_\infty |x - y|,$$

which then gives (4.15) for $p = \infty$ and $\varphi \in C_b^\infty(\mathbb{R}^d)$. For general $\varphi \in L^\infty(\mathbb{R}^d)$, it follows by a standard approximation. For $p \in [1, \infty)$, it is similar. \square

5. EXISTENCE OF GLOBAL SOLUTIONS FOR FNSE WITH LARGE VISCOSITY

In this section, we prove the global existence of $\mathbb{W}^{1,p}$ -solution for stochastic system (3.22) in the case of large viscosity and periodic boundary. Let \mathbb{T}^d be the d -dimensional torus. Below, we shall work in the Sobolev space $\mathbb{W}^{1,p}(\mathbb{T}^d; \mathbb{R}^d)$ with vanishing mean denoted by $\mathbb{W}_0^{1,p}(\mathbb{T}^d; \mathbb{R}^d)$. Thus, by Poincaré's inequality, for any $p > 1$,

$$\|u\|_p \leq C \|\nabla u\|_p,$$

and an equivalent norm in $\mathbb{W}^{1,p}(\mathbb{T}^d; \mathbb{R}^d)$ is thus given by

$$\|u\|_{1,p} \simeq \|\nabla u\|_p.$$

Below, we shall use $\|\nabla u\|_p$ as the norm of $\mathbb{W}_0^{1,p}(\mathbb{T}^d; \mathbb{R}^d)$.

Theorem 5.1. *For any divergence free $u_0 \in \mathbb{W}_0^{1,p}(\mathbb{T}^d; \mathbb{R}^d)$ with $p > \frac{2d}{\alpha}$, there exist $\nu = \nu(\|u_0\|_{1,p}) \geq 1$ sufficiently large and a unique pair of (u, X) with $u \in C(\mathbb{R}_-, \mathbb{W}_0^{1,p}(\mathbb{T}^d; \mathbb{R}^d))$ solving stochastic system (3.22).*

Proof. By Theorem 3.6, there exist a time $T = T(\|\nabla u_0\|_p) \in [-1, 0)$ independent of $\nu \geq 1$ and a unique pair of (u, X) with $u \in C([T, 0], \mathbb{W}_0^{1,p})$ and

$$\|\nabla u_t\|_p \leq 3C_0 \|\nabla u_0\|_p$$

solving stochastic system (3.22). Our aim is to prove that for some $T_* = T_*(\|\nabla u_0\|_p) \in [T, 0)$ and ν large enough,

$$\|\nabla u_{T_*}\|_p \leq \|\nabla u_0\|_p. \quad (5.1)$$

After proving this estimate, one can invoke the standard bootstrap argument to obtain the global solution.

By (3.9), we have

$$\begin{aligned} \partial_t u_t &= \mathbf{P}\mathbb{E}[(\nabla^t X_{t,0} - \mathbb{I}) \cdot \nabla u_0 \circ X_{t,0} \cdot \partial_t X_{t,0}] + \mathbf{P}\partial_i \mathbb{E}[u_0 \circ X_{t,0}] \\ &\quad + \mathbf{P}\mathbb{E}[\nabla^t X_{t,0} \cdot \nabla^t u_0 \circ X_{t,0} \cdot \partial_i(X_{t,0} - x)] + \mathbf{P}\mathbb{E}[\nabla(u_0^i \circ X_{t,0})]. \end{aligned}$$

Noticing that

$$\nabla X_{t,s}(x) - \mathbb{I} = \int_t^s \nabla u_r(X_{t,r}(x)) \cdot \nabla X_{t,r}(x) dr,$$

by Hölder's inequality, Theorem 3.1 and (3.6), we have for any $\gamma \in (1, \frac{p\alpha}{2d})$ and $x \in \mathbb{R}^d$,

$$\begin{aligned} \mathbb{E}|\nabla^t X_{t,0}(x) - \mathbb{I}|^2 &\leq \mathbb{E} \left[\int_t^0 |\nabla u_r(X_{t,r}(x))|^2 dr \int_t^0 |\nabla X_{t,r}(x)|^2 dr \right] \\ &\leq |t|^2 \left[\mathbb{E} \int_t^0 |\nabla u_r(X_{t,r}(x))|^{2\gamma} dr \right]^{1/\gamma} \left[\mathbb{E} \int_t^0 |\nabla X_{t,r}(x)|^{2\gamma^*} dr \right]^{1/\gamma^*} \leq C|t|^2, \end{aligned}$$

where $\gamma^* = \frac{\gamma}{\gamma-1}$ and C is independent of x and t , and depends on $\|\nabla u_0\|_p$. Hence, by Hölder's inequality,

$$\|\mathbb{E}[(\nabla^t X_{t,0} - \mathbb{I}) \cdot \nabla u_0 \circ X_{t,0} \cdot \partial_t X_{t,0}]\|_p$$

$$\begin{aligned}
&\leq \| |\nabla X_{t,0} - \mathbb{I}| \|_{L^2(\Omega)} \cdot \|\nabla u_0 \circ X_{t,0}\|_{L^p(\Omega)} \cdot \|\partial_i X_{t,0}\|_{L^{2p/(p-2)}(\Omega)} \|p \\
&\leq C \sup_{x \in \mathbb{R}^d} \|\nabla X_{t,0}(x) - \mathbb{I}\|_{L^2(\Omega)} \cdot \sup_{x \in \mathbb{R}^d} \|\partial_i X_{t,0}(x)\|_{L^{2p/(p-2)}(\Omega)} \cdot \|\nabla u_0\|_p \\
&\leq C|t| \cdot \|\nabla u_0\|_p.
\end{aligned}$$

Similarly,

$$\|\mathbb{E}[\nabla^t X_{t,0} \cdot \nabla^t u_0 \circ X_{t,0} \cdot \partial_i(X_{t,0} - x)]\|_p \leq C|t| \cdot \|\nabla u_0\|_p.$$

On the other hand, by (4.15), we have

$$\|\partial_i \mathbb{E}[u_0 \circ X_{t,0}]\|_p + \|\nabla \mathbb{E}[u_0^i \circ X_{t,0}]\|_p \leq C(\nu|t|)^{-1/\alpha} \|u_0\|_p \leq C(\nu|t|)^{-1/\alpha} \|\nabla u_0\|_p.$$

Combining the above calculations, we obtain that

$$\|\nabla u_t\|_p \leq C(|t| + (\nu|t|)^{-1/\alpha}) \|\nabla u_0\|_p,$$

which then gives (5.1) by first letting t small enough and then ν large enough. The proof is thus complete. \square

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